Our aim is to order the vertices so that each has at most $k-1$ lower-indexed neighbors; greedy coloring for such an ordering yields the bound.

When G is not k-regular, we can choose a vertex of degree less than k as v_n. Since G is connected, we can grow a spanning tree of G from v_n, assigning indices in decreasing order as we reach vertices. Each vertex other than v_n in the resulting ordering $v_1, \ldots, v_n$ has a higher-indexed neighbor along the path to v_n in the tree. Hence each vertex has at most $k-1$ lower-indexed neighbors, and the greedy coloring uses at most k colors.

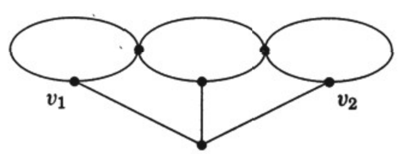
In the remaining case, G is k-regular. Suppose first that G has a cut-vertex x, and let G' be a subgraph consisting of a component of $G-x$ together with its edges to x. The degree of x in G' is less than k, so the method above provides a proper k-coloring of G'. By permuting the names of colors in the subgraphs resulting in this way from components of $G-x$, we can make the colorings agree on x to complete a proper k-coloring of G.

We may thus assume that G is 2-connected. In every vertex ordering, the last vertex has k earlier neighbors. The greedy coloring idea may still work if we arrange that two neighbors of v_n get the same color.

In particular, suppose that some vertex v_n has neighbors v_1, v_2 such that $v_1 \not\leftrightarrow v_2$ and $G-\{v_1, v_2\}$ is connected. In this case, we index the vertices of a spanning tree of $G-\{v_1, v_2\}$ using $3, \ldots, n$ such that labels increase along paths to the root v_n. As before, each vertex before v_n has at most $k-1$ lower indexed neighbors. The greedy coloring also uses at most $k-1$ colors on neighbors of v_n, since v_1 and v_2 receive the same color.

Hence it suffices to show that every 2-connected k-regular graph with $k \ge 3$ has such a triple v_1, v_2, v_n. Choose a vertex x. If $\kappa(G-x) \ge 2$, let v_1 be x and let v_2 be a vertex with distance 2 from x. Such a vertex v_2 exists because G is regular and is not a complete graph; let v_n be a common neighbor of v_1 and v_2.

If $\kappa(G-x) = 1$, let $v_n = x$. Since G has no cut-vertex, x has a neighbor in every leaf block of $G-x$. Neighbors v_1, v_2 of x in two such blocks are nonadjacent. Also, $G-\{x, v_1, v_2\}$ is connected, since blocks have no cut-vertices. Since $k \ge 3$, vertex x has another neighbor, and $G-\{v_1, v_2\}$ is connected. ■



5.1.23.* Remark. The bound $\chi(G) \le \Delta(G)$ can be improved when G has no large clique (Exercise 50). Brooks' Theorem implies that the complete graphs and odd cycles are the only $k-1$-regular k-critical graphs (Exercise 47). Gallai

[1963b] strengthened this by proving that in the subgraph of a k-critical graph induced by the vertices of degree $k-1$, every block is a clique or an odd cycle.

Brooks' Theorem states that $\chi(G) \le \Delta(G)$ whenever $3 \le \omega(G) \le \Delta(G)$. Borodin and Kostochka [1977] conjectured that $\omega(G) < \Delta(G)$ implies $\chi(G) < \Delta(G)$ if $\Delta(G) \ge 9$ (examples show that the condition $\Delta(G) \ge 9$ is needed). Reed [1999] proved that this is true when $\Delta(G) \ge 10^{14}$.

Reed [1998] also conjectured that the chromatic number is bounded by the average of the trivial upper and lower bounds; that is, $\chi(G) \le \lceil \frac{\Delta(G)+1+\omega(G)}{2} \rceil$. ■

Because the idea of partitioning to satisfy constraints is so fundamental, there are many, many variations and generalizations of graph coloring. In Chapter 7 we consider coloring the edges of a graph. Sticking to vertices, we could allow color classes to induce subgraphs other than independent sets ("generalized coloring"—Exercises 49–53). We could restrict the colors allowed to be used on each vertex ("list coloring"—Section 8.4). We could ask questions involving numerical values of the colors (Exercise 54). We have only touched the tip of the iceberg on coloring problems.

EXERCISES

5.1.1. (–) Compute the clique number, the independence number, and the chromatic number of the graph below. Does either bound in Proposition 5.1.7 prove optimality for some proper coloring? Is the graph color-critical?

5.1.2. (–) Prove that the chromatic number of a graph equals the maximum of the chromatic numbers of its components.

5.1.3. (–) Let $G_1, \ldots, G_k$ be the blocks of a graph G. Prove that $\chi(G) = \max_i \chi(G_i)$.

5.1.4. (–) Exhibit a graph G with a vertex v so that $\chi(G-v) < \chi(G)$ and $\chi(\overline{G}-v) < \chi(\overline{G})$.

5.1.5. (–) Given graphs G and H, prove that $\chi(G+H) = \max\{\chi(G), \chi(H)\}$ and that $\chi(G \vee H) = \chi(G) + \chi(H)$.

5.1.6. (–) Suppose that $\chi(G) = \omega(G) + 1$, as in Example 5.1.8. Let $H_1 = G$ and $H_k = H_{k-1} \vee G$ for $k > 1$. Prove that $\chi(H)_k = \omega(H)_k + k$.

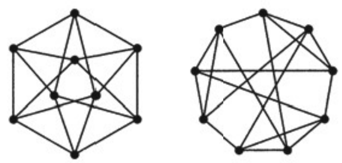
5.1.7. (–) Construct a graph G that is neither a clique nor an odd cycle but has a vertex ordering relative to which greedy coloring uses $\Delta(G)+1$ colors.

5.1.8. (–) Prove that $\max_{H \subseteq G} \delta(H) \le \Delta(G)$ to explain why Theorem 5.1.19 is better than Proposition 5.1.13. Determine all graphs G such that $\max_{H \subseteq G} \delta(H) = \Delta(G)$.

5.1.9. (–) Draw the graph $K_{1,3} \square P_3$ and exhibit an optimal coloring of it. Draw $C_5 \square C_5$ and find a proper 3-coloring of it with color classes of sizes 9, 8, 8.

5.1.10. (−) Prove that $G \square H$ decomposes into $n(G)$ copies of H and $n(H)$ copies of G.

5.1.11. (−) Prove that each graph below is isomorphic to $C_3 \square C_3$.



5.1.12. (−) Prove or disprove: Every k-chromatic graph G has a proper k-coloring in which some color class has $\alpha(G)$ vertices.

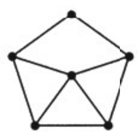
5.1.13. (−) Prove or disprove: If $G = F \cup H$, then $\chi(G) \le \chi(F) + \chi(H)$.

5.1.14. (−) Prove or disprove: For every graph G, $\chi(G) \le n(G) - \alpha(G) + 1$.

5.1.15. (−) Prove or disprove: If G is a connected graph, then $\chi(G) \le 1 + a(G)$, where $a(G)$ is the average of the vertex degrees in G.

5.1.16. (−) Use Theorem 5.1.21 to prove that every tournament has a spanning path. (Rédei [1934])

5.1.17. (−) Use Lemma 5.1.18 to prove that $\chi(G) \le 4$ for the graph G below.



5.1.18. (−) Determine the number of colors needed to label $V(K_n)$ such that each color class induces a subgraph with maximum degree at most k.

5.1.19. (−) Find the error in the false argument below for Brooks' Theorem (Theorem 5.1.22).

"We use induction on $n(G)$; the statement holds when $n(G) = 1$. For the induction step, suppose that G is not a complete graph or an odd cycle. Since $\kappa(G) \le \delta(G)$, the graph G has a separating set S of size at most $\Delta(G)$. Let $G_1, \dots, G_m$ be the components of $G - S$, and let $H_i = G[V(G_i) \cup S]$. By the induction hypothesis, each H_i is $\Delta(G)$-colorable. Permute the names of the colors used on these subgraphs to agree on S. This yields a proper $\Delta(G)$-coloring of G."

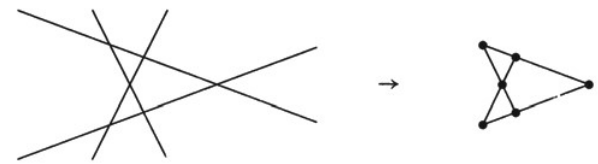
• • • • •

5.1.20. (!) Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \le 5$.

5.1.21. Suppose that every edge of a graph G appears in at most one cycle. Prove that every block of G is an edge, a cycle, or an isolated vertex. Use this to prove that $\chi(G) \le 3$.

5.1.22. (!) Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \le 3$. (Hint: This

can be solved by using the Szekeres–Wilf Theorem or by using greedy coloring with an appropriate vertex ordering. Comment: The conclusion may fail when three lines are allowed to share a point.) (H. Sachs)



5.1.23. (!) Place n points on a circle, where $n \ge k(k+1)$. Let $G_{n,k}$ be the $2k$-regular graph obtained by joining each point to the k nearest points in each direction on the circle. For example, $G_{n,1} = C_n$, and $G_{7,2}$ appears below. Prove that $\chi(G_{n,k}) = k+1$ if $k+1$ divides n and $\chi(G_{n,k}) = k+2$ if $k+1$ does not divide n. Prove that the lower bound on n cannot be weakened, by proving that $\chi(G_{k(k+1)-1,k}) > k+2$ if $k \ge 2$.

5.1.24. (+) Let G be any 20-regular graph with 360 vertices formed in the following way. The vertices are evenly-spaced around a circle. Vertices separated by 1 or 2 degrees are nonadjacent. Vertices separated by 3, 4, 5 or 6 degrees are adjacent. No information is given about other adjacencies (except that G is 20-regular). Prove that $\chi(G) \le 19$. (Hint: Color successive vertices in order around the circle.) (Pritikin)

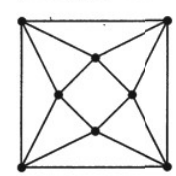
5.1.25. (+) Let G be the **unit-distance graph** in the plane; $V(G) = \mathbb{R}^2$, and two points are adjacent if their Euclidean distance is 1 (this is an infinite graph). Prove that $4 \le \chi(G) \le 7$. (Hint: For the upper bound, present an explicit coloring by regions, paying attention to the boundaries.) (Hadwiger [1945, 1961], Moser–Moser [1961])

5.1.26. Given finite sets $S_1, \dots, S_m$, let $U = S_1 \times \cdots \times S_m$. Define a graph G with vertex set U by putting $u \leftrightarrow v$ if and only if u and v differ in every coordinate. Determine $\chi(G)$.

5.1.27. Let H be the complement of the graph in Exercise 5.1.26. Determine $\chi(H)$.

5.1.28. Consider a traffic signal controlled by two switches, each of which can be set in n positions. For each setting of the switches, the traffic signal shows one of its n possible colors. Whenever the setting of *both* switches changes, the color changes. Prove that the color shown is determined by the position of one of the switches. Interpret this in terms of the chromatic number of some graph. (Greenwell–Lovász [1974])

5.1.29. For the graph G below, compute $\chi(G)$ and find a $\chi(G)$-critical subgraph.



5.1.30. (+) Let $S = \binom{[n]}{2}$ denote the collection of 2-sets of the n-element set $[n]$. Define the graph G_n by $V(G_n) = S$ and $E(G_n) = \{(ij, jk)\colon 1 \le i < j < k \le n\}$ (disjoint pairs, for example, are nonadjacent). Prove that $\chi(G_n) = \lceil \lg n \rceil$. (Hint: Prove that G_n is r-colorable if and only if $[r]$ has at least n distinct subsets. Comment: G_n is called the **shift graph** of K_n.) (attributed to A. Hajnal)

5.1.31. (!) Prove that a graph G is m-colorable if and only if $\alpha(G \,\square\, K_m) \ge n(G)$. (Berge [1973, p379–80])

5.1.32. (!) Prove that a graph G is 2^k-colorable if and only if G is the union of k bipartite graphs. (Hint: This generalizes Theorem 1.2.23.)

5.1.33. (!) Prove that every graph G has a vertex ordering relative to which greedy coloring uses $\chi(G)$ colors.

5.1.34. (!) For all $k \in \mathbb{N}$, construct a tree T_k with maximum degree k and an ordering σ of $V(T_k)$ such that greedy coloring relative to the ordering σ uses $k+1$ colors. (Hint: Use induction and construct the tree and ordering simultaneously. Comment: This result shows that the performance ratio of greedy coloring to optimal coloring can be as bad as $(\Delta(G)+1)/2$.) (Bean [1976])

5.1.35. Let G be a graph having no induced subgraph isomorphic to P_4. Prove that for every vertex ordering, greedy coloring produces an optimal coloring of G. (Hint: Suppose that the algorithm uses k colors for the ordering $v_1, \dots, v_n$, and let i be the smallest integer such that G has a clique consisting of vertices assigned colors i through k in this coloring. Prove that $i = 1$. Comment: P_4-free graphs are also called **cographs**.)

5.1.36. Given a vertex ordering $\sigma = v_1, \dots, v_n$ of a graph G, let $G_i = G[\{v_1, \dots, v_i\}]$ and $f(\sigma) = 1 + \max_i d_{G_i}(v_i)$. Greedy coloring relative to σ yields $\chi(G) \le f(\sigma)$. Define σ^* by letting v_n be a minimum degree vertex of G and letting v_i for $i < n$ be a minimum degree vertex of $G - \{v_{i+1}, \dots, v_n\}$. Show that $f(\sigma^*) = 1 + \max_{H \subseteq G} \delta(H)$, and thus that σ^* minimizes $f(\sigma)$. (Halin [1967], Matula [1968], Finck–Sachs [1969], Lick–White [1970])

5.1.37. Prove that $V(G)$ can be partitioned into $1 + \max_{H \subseteq G} \delta(H)/r$ classes such that every subgraph whose vertices lie in a single class has a vertex of degree less than r. (Hint: Consider ordering σ^* of Exercise 5.1.36. Comment: This generalizes Theorem 5.1.19. See also Chartrand–Kronk [1969] when $r = 2$.)

5.1.38. (!) Prove that $\chi(G) = \omega(G)$ when $\overline{G}$ is bipartite. (Hint: Phrase the claim in terms of $\overline{G}$ and apply results on bipartite graphs.)

5.1.39. (!) Prove that every k-chromatic graph has at least $\binom{k}{2}$ edges. Use this to prove that if G is the union of m complete graphs of order m, then $\chi(G) \le 1 + m\sqrt{m-1}$. (Comment: This bound is near tight, but the Erdős–Faber–Lovász Conjecture (see Erdős [1981]) asserts that $\chi(G) = m$ when the complete graphs are pairwise edge-disjoint.)

5.1.40. Prove that $\chi(G) \cdot \chi(\overline{G}) \ge n(G)$, use this to prove that $\chi(G) + \chi(\overline{G}) \ge 2\sqrt{n(G)}$, and provide a construction achieving these bounds whenever $\sqrt{n(G)}$ is an integer. (Nordhaus–Gaddum [1956], Finck [1968])

5.1.41. (!) Prove that $\chi(G) + \chi(\overline{G}) \le n(G) + 1$. (Hint: Use induction on $n(G)$.) (Nordhaus–Gaddum [1956])

5.1.42. (!) *Looseness of* $\chi(G) \ge n(G)/\alpha(G)$. Let G be an n-vertex graph, and let $c = (n+1)/\alpha(G)$. Use Exercise 5.1.41 to prove that $\chi(G) \cdot \chi(\overline{G}) \le (n+1)^2/4$, and use this to prove that $\chi(G) \le c(n+1)/4$. For each odd n, construct a graph such that $\chi(G) = c(n+1)/4$. (Nordhaus–Gaddum [1956], Finck [1968])

5.1.43. (!) *Paths and chromatic number in digraphs.*

a) Let $G = F \cup H$. Prove that $\chi(G) \le \chi(F)\chi(H)$.

b) Consider an orientation D of G and a function $f\colon V(G) \to \mathbb{R}$. Use part (a) and Theorem 5.1.21 to prove that if $\chi(G) > rs$, then D has a path $u_0 \to \cdots \to u_r$ with $f(u_0) \le \cdots \le f(u_r)$ or a path $v_0 \to \cdots \to v_s$ with $f(v_0) > \cdots > f(v_s)$.

c) Use part (b) to prove that every sequence of $rs+1$ distinct real numbers has an increasing subsequence of size $r+1$ or a decreasing subsequence of size $s+1$. (Erdős–Szekeres [1935])

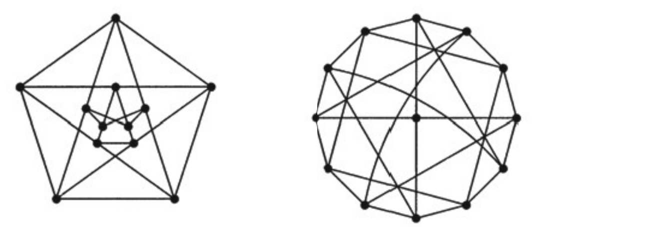
5.1.44. (!) *Minty's Theorem* (Minty [1962]). An **acyclic orientation** of a loopless graph is an orientation having no cycle. For each acyclic orientation D of G, let $r(D) = \max_C \lceil a/b \rceil$, where C is a cycle in G and a, b count the edges of C that are forward in D or backward in D, respectively. Fix a vertex $x \in V(G)$, and let W be a walk in G beginning at x. Let $g(W) = a - b \cdot r(D)$, where a is the number of steps along W that are forward edges in D and b is the number that are backward in D. For each $y \in V(G)$, let $g(y)$ be the maximum of $g(W)$ such that W is an x, y-walk (assume that G is connected).

a) Prove that $g(y)$ is finite and thus well-defined, and use $g(y)$ to obtain a proper $1 + r(D)$-coloring of G. Thus G is $1 + r(D)$-colorable.

b) Prove that $\chi(G) = \min_{D \in \mathbf{D}}$, where $\mathbf{D}$ is the set of acyclic orientations of G.

5.1.45. (+) Use Minty's Theorem (Exercise 5.1.44) to prove Theorem 5.1.21. (Hint: Prove that $l(D)$ is maximized by some acyclic orientation of G.)

5.1.46. (+) Prove that the 4-regular triangle-free graphs below are 4-chromatic. (Hint: Consider the maximum independent sets. Comment: Chvátal [1970] showed that the graph on the left is the smallest triangle-free 4-regular 4-chromatic graph.)



5.1.47. (!) Prove that Brooks' Theorem is equivalent to the following statement: every $k-1$-regular k-critical graph is a complete graph or an odd cycle.

5.1.48. Let G be a simple graph with n vertices and m edges and maximum degree at most 3. Suppose that no component of G is a complete graph on 4 vertices. Prove that G contains a bipartite subgraph with at least $m - n/3$ edges. (Hint: Apply Brooks' Theorem, and then show how to delete a few edges to change a proper 3-coloring of G into a proper 2-coloring of a large subgraph of G.)

5.1.49. (−) Prove that the Petersen graph can be 2-colored so that the subgraph induced by each color class consists of isolated edges and vertices.

5.1.50. (!) *Improvement of Brooks' Theorem.*

a) Given a graph G, let $k_1, \dots, k_t$ be nonnegative integers with $\sum k_i \ge \Delta(G) - t + 1$. Prove that $V(G)$ can be partitioned into sets $V_1, \dots, V_t$ so that for each i, the subgraph G_i induced by V_i has maximum degree at most k_i. (Hint: Prove that the partition minimizing $\sum e(G_i)/k_i$ has the desired property.) (Lovász [1966])

b) For $4 \le r \le \Delta(G)+1$, use part (a) to prove that $\chi(G) \le \left\lceil \frac{r-1}{r}(\Delta(G)+1) \right\rceil$ when G has no r-clique. (Borodin–Kostochka [1977], Catlin [1978], Lawrence [1978])

5.1.51. (!) Let G be an k-colorable graph, and let P be a set of vertices in G such that $d(x, y) \ge 4$ whenever $x, y \in P$. Prove that every coloring of P with colors from $[k+1]$ extends to a proper $k+1$ coloring of G. (Albertson–Moore [1999])

5.1.52. Prove that every graph G can be $\lceil(\Delta(G)+1)/j\rceil$-colored so that each color class induces a subgraph having no j-edge-connected subgraph. For $j > 1$, prove that no smaller number of classes suffices when G is a j-regular j-edge-connected graph or is a complete graph with order congruent to 1 modulo j. (Comment: For $j = 1$, the restriction reduces to ordinary proper coloring.) (Matula [1973])

5.1.53. (+) Let $G_{n,k}$ be the $2k$-regular graph of Exercise 5.1.23. For $k \le 4$, determine the values of n such that $G_{n,k}$ can be 2-colored so that each color class induces a subgraph with maximum degree at most k. (Weaver–West [1994])

5.1.54. Let f be a proper coloring of a graph G in which the colors are natural numbers. The **color sum** is $\sum_{v \in V(G)} f(v)$. Minimizing the color sum may require using more than $\chi(G)$ colors. In the tree below, for example, the best proper 2-coloring has color sum 12, while there is a proper 3-coloring with color sum 11. Construct a sequence of trees in which the kth tree T_k use k colors in a proper coloring that minimizes the color sum. (Kubicka–Schwenk [1989])

5.1.55. (+) *Chromatic number is bounded by one plus longest odd cycle length.*

a) Let G be a 2-connected nonbipartite graph containing an even cycle C. Prove that there exist vertices x, y on C and an x, y-path P internally disjoint from C such that $d_C(x, y) \ne d_P(x, y) \bmod 2$.

b) Let G be a simple graph with no odd cycle having length at least $2k+1$. Prove that if $\delta(G) \ge 2k$, then G has a cycle of length at least $4k$. (Hint: Consider the neighbors of an endpoint of a maximal path.)

c) Let G be a 2-connected nonbipartite graph with no odd cycle longer than $2k-1$. Prove that $\chi(G) \le 2k$. (Erdős–Hajnal [1966])

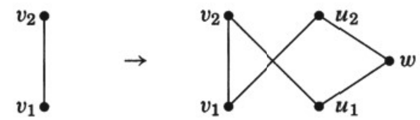
5.2. Structure of k-chromatic Graphs

We have observed that $\chi(H) \ge \omega(H)$ for all H. When equality holds in this bound for G and all its induced subgraphs (as for interval graphs), we say that G is **perfect**; we discuss such graphs in Sections 5.3 and 8.1. Our concern with the bound $\chi(G) \ge \omega(G)$ in this section is how *bad* it can be. Almost always $\chi(G)$ is much larger than $\omega(G)$, in a sense discussed precisely in Section 8.5. (The average values of $\omega(G)$, $\alpha(G)$, and $\chi(G)$ over all graphs with vertex set $[n]$ are very close to $2\lg n$, $2\lg n$, and $n/(2\lg n)$, respectively. Hence $\omega(G)$ is generally a bad lower bound on $\chi(G)$, and $n/\alpha(G)$ is generally a good lower bound.)

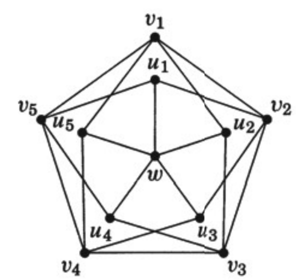
GRAPHS WITH LARGE CHROMATIC NUMBER

The bound $\chi(G) \ge \omega(G)$ can be tight, but it can also be very loose. There have been many constructions of graphs without triangles that have arbitrarily large chromatic number. We present one such construction here; others appear in Exercises 12–13.

5.2.1. Definition. From a simple graph G, **Mycielski's construction** produces a simple graph G' containing G. Beginning with G having vertex set $\{v_1, \dots, v_n\}$, add vertices $U = \{u_1, \dots, u_n\}$ and one more vertex w. Add edges to make u_i adjacent to all of $N_G(v_i)$, and finally let $N(w) = U$.



5.2.2. Example. From the 2-chromatic graph K_2, one iteration of Mycielski's construction yields the 3-chromatic graph C_5, as shown above. Below we apply the construction to C_5, producing the 4-chromatic **Grötzsch graph**. ■



5.2.3. Theorem. (Mycielski [1955]) From a k-chromatic triangle-free graph G, Mycielski's construction produces a $k+1$-chromatic triangle-free graph G'.

Proof: Let $V(G) = \{v_1, \dots, v_n\}$, and let G' be the graph produced from it by Mycielski's construction. Let $u_1, \dots, u_n$ be the copies of $v_1, \dots, v_n$, with w the additional vertex. Let $U = \{u_1, \dots, u_n\}$.

By construction, U is an independent set in G'. Hence the other vertices of any triangle containing u_i belong to $V(G)$ and are neighbors of v_i. This would complete a triangle in G, which can't exist. We conclude that G' is triangle-free.

A proper k-coloring f of G extends to a proper $k+1$-coloring of G' by setting $f(u_i) = f(v_i)$ and $f(w) = k+1$; hence $\chi(G') \le \chi(G)+1$. We prove equality by showing that $\chi(G) < \chi(G')$. To prove this we consider any proper coloring of G' and obtain from it a proper coloring of G using fewer colors.

Let g be a proper k-coloring of G' By changing the names of colors, we may assume that $g(w) = k$. This restricts g to $\{1, \dots, k-1\}$ on U. On $V(G)$, it may

$k(n(G) - n(H') + 1)$ edges. Since $\delta(G) \geq 2k$, every 1-vertex subgraph of G is such a subgraph. Since such subgraphs exist, we may choose H to be a maximal subgraph with this property.

Let S be the set of vertices outside H with neighbors in H, and let $G' = G[S]$. We need only show that $\delta(G') \geq k$. Each $x \in V(G')$ has a neighbor $y \in V(H)$. In $G \cdot (H \cup xy)$, the edges incident to x in G' collapse onto edges from $V(G')$ to H that appear in $G \cdot H$, and the edge xy contracts. Hence $e(G \cdot H) - e(G \cdot (H \cup xy)) = d_{G'}(x) + 1$. By the choice of H, this difference is more than k, and hence $\delta(G') \geq k$. ■

5.2.23.* Theorem. (Mader [1967], see Thomassen [1988]) If F and G are simple graphs with $e(F) = m$ and $\delta(F) \geq 1$, then $\delta(G) \geq 2^m$ implies that G contains a subdivision of F.

Proof: We use induction on m. The claim is trivial for $m \leq 1$. Consider $m \geq 2$. By Lemma 5.2.22, we may choose disjoint subgraphs H and G' in G such that H is connected, $\delta(G') \geq 2^{m-1}$, and every vertex of G' has a neighbor in H.

If F has an edge $e = xy$ such that $\delta(F - e) \geq 1$, then the induction hypothesis yields a subdivision J of $F - e$ in G'. A path through H can be added between the vertices of J representing x and y to complete a subdivision of F.

If $\delta(F - e) = 0$ for all $e \in E(F)$, then every edge of F is incident to a leaf. Now F is a forest of stars, and $\delta(G) \geq 2^m \geq 2m$ allows us to find F itself in G; we leave this claim to Exercise 42. ■

5.2.24.* Remark. The case when F is a complete graph remains of particular interest. Let $f(k)$ be the minimum d such that every graph with minimum degree at least d contains a K_k-subdivision. Theorem 5.2.23 yields $f(k) \leq 2^{\binom{k}{2}}$. Komlós–Szemerédi [1996] and Bollobás–Thomason [1998] proved that $f(k) < ck^2$ for some constant c (the latter shows $c \leq 256$). Since $K_{m,m-1}$ has no K_{2k}-subdivision when $m = k(k+1)/2$ (Exercise 41), we have $f(k) > k^2/8$.

Exercise 38 yields $f(4) = 3$. Furthermore, $f(5) = 6$. The icosahedron (Exercise 7.3.8) yields $f(5) \geq 6$, since this graph is 5-regular and has no K_5-subdivision. On the other hand, Mader [1998] proved Dirac's conjecture [1964] that every n-vertex graph with at least $3n - 5$ edges contains a K_5-subdivision. By the degree-sum formula, $\delta(G) \geq 6$ yields at least $3n$ edges; hence $f(5) \leq 6$.

Finally, we note that Scott [1997] proved a subdivision version of the Gyárfas–Sumner Conjecture (Remark 5.2.4) for each tree T and integer k: If G has with no k-clique but $\chi(G)$ is *sufficiently* large, then G contains a subdivision of T as an *induced* subgraph. ■

EXERCISES

5.2.1. (–) Let G be a graph such that $\chi(G - x - y) = \chi(G) - 2$ for all pairs x, y of distinct vertices. Prove that G is a complete graph. (Comment: Lovász conjectured that the conclusion also holds when the condition is imposed only on pairs of adjacent vertices.)

5.2.2. (–) Prove that a simple graph is a complete multipartite graph if and only if it has no 3-vertex induced subgraph with one edge.

5.2.3. (–) The results below imply that there is no k-critical graph with $k + 1$ vertices.

a) Let x and y be vertices in a k-critical graph G. Prove that $N(x) \subseteq N(y)$ is impossible. Conclude that no k-critical graph has $k + 1$ vertices.

b) Prove that $\chi(G \vee H) = \chi(G) + \chi(H)$, and that $G \vee H$ is color-critical if and only if both G and H are color-critical. Conclude that $C_5 \vee K_{k-3}$, with $k + 2$ vertices, is k-critical.

5.2.4. For $n \in \mathbb{N}$, let G be the graph with vertex set $\{v_0, \dots, v_{3n}\}$ defined by $v_i \leftrightarrow v_j$ if and only if $|i - j| \leq 2$ and $i + j$ is not divisible by 6.

a) Determine the blocks of G.

b) Prove that adding the edge $v_0 v_{3n}$ to G creates a 4-critical graph.

5.2.5. (–) Find a subdivision of K_4 in the Grötzsch graph (Example 5.2.2).

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5.2.6. Determine the minimum number of edges in a connected n-vertex graph with chromatic number k. (Hint: Consider a k-critical subgraph.) (Eršov–Kožuhin [1962]—see Bhasker–Samad–West [1994] for higher connectivity.)

5.2.7. (!) Given an optimal coloring of a k-chromatic graph, prove that for each color i there is a vertex with color i that is adjacent to vertices of the other $k - 1$ colors.

5.2.8. Use properties of color-critical graphs to prove Proposition 5.1.14 again: $\chi(G) \leq 1 + \max_i \min\{d_i, i - 1\}$, where $d_1 \geq \cdots \geq d_n$ are the vertex degrees in G.

5.2.9. (!) Prove that if G is a color-critical graph, then the graph G' generated from it by applying Mycielski's construction is also color-critical.

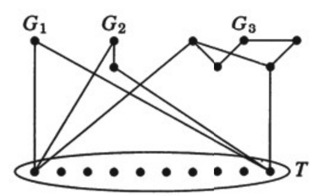
5.2.10. Given a graph G with vertex set $v_1, \dots, v_n$, let G' be the graph generated from G by Mycielski's construction. Let H be a subgraph of G. Let G'' be the graph obtained from G' by adding the edges $\{u_i u_j \colon v_i v_j \in E(H)\}$. Prove that $\chi(G'') = \chi(G) + 1$ and that $\omega(G'') = \max\{\omega(G), \omega(H) + 1\}$. (Pritikin)

5.2.11. (!) Prove that if G has no induced $2K_2$, then $\chi(G) \leq \binom{\omega(G)+1}{2}$. (Hint: Use a maximum clique to define a collection of $\binom{\omega(G)}{2} + \omega(G)$ independent sets that cover the vertices. Comment: This is a special case of the Gyárfás–Sumner Conjecture—Remark 5.2.4) (Wagon [1980])

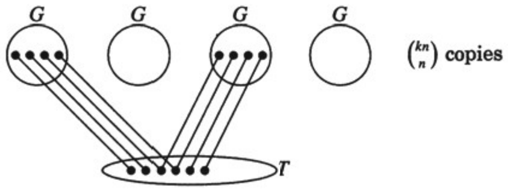
5.2.12. (!) Let $G_1 = K_1$. For $k > 1$, construct G_k as follows. To the disjoint union $G_1 + \cdots + G_{k-1}$, and add an independent set T of size $\prod_{i=1}^{k-1} n(G_i)$. For each choice of $(v_1, \dots, v_{k-1})$ in $V(G_1) \times \cdots \times V(G_{k-1})$, let one vertex of T have neighborhood $\{v_1, \dots, v_{k-1}\}$. (In the sketch of G_4 below, neighbors are shown for only two elements of T.)

a) Prove that $\omega(G_k) = 2$ and $\chi(G_k) = k$. (Zykov [1949])

b) Prove that G_k is k-critical. (Schäuble [1969])



5.2.13. (+) Let G be a k-chromatic graph with girth 6 and order n. Construct G' as follows. Let T be an independent set of kn new vertices. Take $\binom{kn}{n}$ pairwise disjoint copies of G, one for each way to choose an n-set $S \subset T$. Add a matching between each copy of G and its corresponding n-set S. Prove that the resulting graph has chromatic number $k+1$ and girth 6. (Comment: Since C_6 is 2-chromatic with girth 6, the process can start and these graphs exist.) (Blanche Descartes [1947, 1954])



5.2.14. *Chromatic number and cycle lengths.*

a) Let v be a vertex in a graph G. Among all spanning trees of G, let T be one that maximizes $\sum_{u\in V(G)} d_T(u,v)$. Prove that every edge of G joins vertices belonging to a path in T starting at v.

b) Prove that if $\chi(G) > k$, then G has a cycle whose length is one more than a multiple of k. (Hint: Use the tree T of part (a) to define a k-coloring of G.) (Tuza)

5.2.15. (!) Prove that a triangle-free graph with n vertices is colorable with $2\sqrt{n}$ colors. (Comment: Thus every k-chromatic triangle-free graph has at least $k^2/4$ vertices.)

5.2.16. (!) Prove that every n-vertex simple graph with no $r+1$-clique has at most $(1-1/r)n^2/2$ edges. (Hint: This can be proved using Turán's Theorem or by induction on r without Turán's Theorem.)

5.2.17. (!) Let G be a simple n-vertex graph with m edges.

a) Prove that $\omega(G) \ge \lceil n^2/(n^2-2m)\rceil$ and that this bound is sharp. (Hint: Use Exercise 5.2.16. Comment: This also yields $\chi(G) \ge \lceil n^2/(n^2-2m)\rceil$.) (Myers–Liu [1972])

b) Prove that $\alpha(G) \ge \lceil n/(d+1)\rceil$, where d is the average vertex degree of G. (Hint: Use part (a).) (Erdős–Gallai [1961])

5.2.18. The Turán graph $T_{n,r}$ (Example 5.2.7) is the complete r-partite graph with b partite sets of size $a+1$ and $r-b$ partite sets of size a, where $a = \lfloor n/r\rfloor$ and $b = n - ra$.

a) Prove that $e(T_{n,r}) = (1-1/r)n^2/2 - b(r-b)/(2r)$.

b) Since $e(G)$ must be an integer, part (a) implies $e(T_{n,r}) \le \lfloor (1-1/r)n^2/2\rfloor$. Determine the smallest r such that strict inequality occurs for some n. For this value of r, determine all n such that $e(T_{n,r}) < \lfloor (1-1/r)n^2/2\rfloor$.

5.2.19. (+) Let $a = \lfloor n/r\rfloor$. Compare the Turán graph $T_{n,r}$ with the graph $\overline{K}_a + K_{n-a}$ to prove directly that $e(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$.

5.2.20. Given positive integers n and k, let $q = \lfloor n/k\rfloor$, $r = n - qk$, $s = \lfloor n/(k+1)\rfloor$, and $t = n - s(k+1)$. Prove that $\binom{q}{2}k + rq \ge \binom{s}{2}(k+1) + ts$. (Hint: Consider the complement of the Turán graph.) (Richter [1993])

5.2.21. Prove that among the n-vertex simple graphs with no $r+1$-clique, the Turán graph $T_{n,r}$ is the *unique* graph having the maximum number of edges. (Hint: Examine the proof of Theorem 5.2.9 more carefully.)

5.2.22. A circular city with diameter four miles will get 18 cellular-phone power stations. Each station has a transmission range of six miles. Prove that no matter where

in the city the stations are placed, at least two will each be able to transmit to at least five others. (Adapted from Bondy–Murty [1976, p115])

5.2.23. (!) *Turán's proof of Turán's Theorem*, including uniqueness (Turán [1941]).

a) Prove that a maximal simple graph with no $r+1$-clique has an r-clique.

b) Prove that $e(T_{n,r}) = \binom{r}{2} + (n-r)(r-1) + e(T_{n-r,r})$.

c) Use parts (a) and (b) to prove Turán's Theorem by induction on n, including the characterization of graphs acheiving the bound.

5.2.24. (+) Let $t_r(n) = e(T_{n,r})$. Let G be a graph with n vertices that has $t_r(n) - k$ edges and at least one $r+1$-clique, where $k \ge 0$. Prove that G has at least $f_r(n) + 1 - k$ cliques of order $r+1$, where $f_r(n) = n - \lceil n/r\rceil - r$. (Hint: Prove that a graph with exactly one $r+1$-clique has at most $t_r(n) - f_r(n)$ edges.) (Erdős [1964], Moon [1965c])

5.2.25. *Partial analogue of Turán's Theorem for $K_{2,m}$.*

a) Prove that if G is simple and $\sum_{v\in V(G)} \binom{d(v)}{2} > (m-1)\binom{n}{2}$, then G contains $K_{2,m}$. (Hint: View $K_{2,m}$ as two vertices with m common neighbors.)

b) Prove that $\sum_{v\in V(G)} \binom{d(v)}{2} \ge e(2e/n - 1)$, where G has e edges.

c) Use parts (a) and (b) to prove that a graph with more than $\frac{1}{2}(m-1)^{1/2}n^{3/2} + n/4$ edges contains $K_{2,m}$.

d) Application: Given n points in the plane, prove that the distance is exactly 1 for at most $\frac{1}{\sqrt{2}}n^{3/2} + n/4$ pairs. (Bondy–Murty [1976, p111–112])

5.2.26. For $n \ge 4$, prove that every n-vertex graph with more than $\frac{1}{2}n\sqrt{n-1}$ edges has girth at most 4. (Hint: Use the methods of Exercise 5.2.25

5.2.27. (+) For $n \ge 6$, prove that the maximum number of edges in a simple m-vertex graph not having two edge-disjoint cycles is $n+3$. (Pósa)

5.2.28. (+) For $n \ge 6$, prove that the maximum number of edges in a simple n-vertex graph not having two disjoint cycles is $3n - 6$. (Pósa)

5.2.29. (!) Let G be a claw-free graph (no induced $K_{1,3}$).

a) Prove that the subgraph induced by the union of any two color classes in a proper coloring of G consists of paths and even cycles.

b) Prove that if G has a proper coloring using exactly k colors, then G has a proper k-coloring where the color classes differ in size by at most one. (Niessen–Kind [2000])

5.2.30. (+) Prove that if G has a proper coloring g in which every color class has at least two vertices, then G has an optimal coloring f in which every color class has at least two vertices. (Hint: If f has a color class with only one vertex, use g to make an alteration in f. The proof can be given algorithmically or by induction on $\chi(G)$.) (Gallai [1963c])

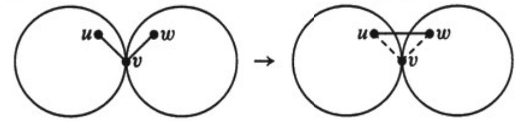
5.2.31. Let G be a connected k-chromatic graph that is not a complete graph or a cycle of length congruent to 3 modulo 6. Prove that every proper k-coloring of G has two vertices of the same color with a common neighbor. (Tomescu)

5.2.32. (!) *The Hajós construction* (Hajós [1961]).

a) Let G and H be k-critical graphs sharing only vertex v, with $vu \in E(G)$ and $vw \in E(H)$. Prove that $(G - vu) \cup (H - vw) \cup uw$ is k-critical.

b) For all $k \ge 3$, use part (a) to obtain a k-critical graph other than K_k.

c) For all $n \ge 4$ except $n = 5$, construct a 4-critical graph with n vertices.



5.2.33. Let G be a k-critical graph having a separating set $S = \{x, y\}$. By Proposition 5.2.18, $x \not\leftrightarrow y$. Prove that G has exactly two S-lobes and that they can be named G_1, G_2 such that $G_1 + xy$ is k-critical and $G_2 \cdot xy$ is k-critical (here $G_2 \cdot xy$ denotes the graph obtained from G_2 by adding xy and then contracting it).

5.2.34. (!) Let G be a 4-critical graph having a separating set S of size 4. Prove that $G[S]$ has at most four edges. (Pritikin)

5.2.35. (+) *Alternative proof that k-critical graphs are $k-1$-edge-connected.*

a) Let G be a k-critical graph, with $k \ge 3$. Prove that for every $e, f \in E(G)$ there is a $k-1$-critical subgraph of G containing e but not f. (Toft [1974])

b) Use part (a) and induction on k to prove Dirac's Theorem that every k-critical graph is $k-1$-edge-connected. (Toft [1974])

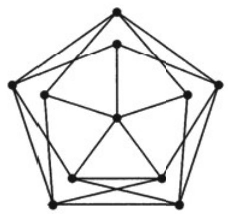
5.2.36. (+) Prove that if G is k-critical and every $k-1$-critical subgraph of G is isomorphic to K_{k-1}, then $G = K_k$ (if $k \ge 4$) (Hint: Use Toft's critical graph lemma—Exercise 5.2.35a.) (Stiebitz [1985])

5.2.37. A graph G is **vertex-color-critical** if $\chi(G - v) < \chi(G)$ for all $v \in V(G)$.

a) Prove that every color-critical graph is vertex-color-critical.

b) Prove that every 3-chromatic vertex-color-critical graph is color-critical.

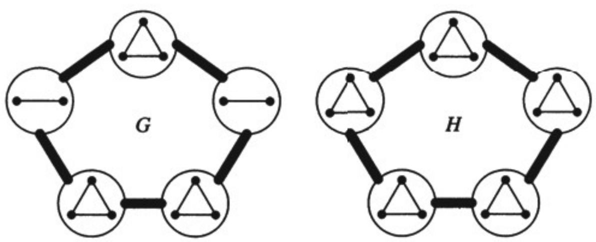
c) Prove that the graph below is vertex-color-critical but not color-critical. (Comment: This is *not* the Grötzsch graph.)



5.2.38. (!) Prove that every simple graph with minimum degree at least 3 contains a K_4-subdivision. (Hint: Prove a stronger result—every nontrivial simple graph with at most one vertex of degree less than 3 contains a K_4-subdivision. The proof of Theorem 5.2.20 already shows that every 3-connected graph contains a K_4-subdivision.) (Dirac [1952a])

5.2.39. (!) Given that $\delta(G) \ge 3$ forces a K_4-subdivision in G, prove that the maximum number of edges in a simple n-vertex graph with no K_4-subdivision is $2n - 3$.

5.2.40. Thick edges below indicate that every vertex in one circle is adjacent to every vertex in the other. Prove that $\chi(G) = 7$ but G has no K_7-subdivision. Prove that $\chi(H) = 8$ but H has no K_8-subdivision. (Catlin [1979])



5.2.41. Let $m = k(k+1)/2$. Prove that $K_{m,m-1}$ has no K_{2k}-subdivision.

5.2.42. (+) Let F be a forest with m edges. Let G be a simple graph such that $\delta(G) \ge m$ and $n(G) \ge n(F)$. Prove that G contains F as a subgraph. (Hint: Delete one leaf from each nontrivial component of F to obtain F'. Let R be the set of neighbors of the deleted vertices. Map R onto an m-set $X \subseteq V(G)$ that minimizes $e(G[X])$. Extend X to a copy of F'. Use Hall's Theorem to show that X can be matched into the remaining vertices to complete a copy of F.) (Brandt [1994])

5.2.43. (+) Let G be a k-chromatic graph. It follows from Lemma 5.1.18 and Proposition 2.1.8 that G contains every k-vertex tree as a subgraph. Strengthen this to a labeled analogue: if f is a proper k-coloring of G and T is a tree with vertex set $\{w_1, \dots, w_k\}$, then there is an adjacency-preserving map $\phi\colon V(T) \to V(G)$ such that $f(\phi(w_i)) = i$ for all i. (Gyárfás–Szemerédi–Tuza [1980], Sumner [1981])

5.2.44. (+) Let G be a k-chromatic graph of girth at least 5. Prove that G contains every k-vertex tree as an induced subgraph. (Gyárfás–Szemerédi–Tuza [1980])

5.3. Enumerative Aspects

Sometimes we can shed light on a hard problem by considering a more general problem. No good algorithm to test existence of a proper k-coloring is known (see Appendix B), but still we can study the number of proper k-colorings (here we fix a particular set of k colors). The chromatic number $\chi(G)$ is the minimum k such that the count is positive; knowing the count for all k would tell us the chromatic number. Birkhoff [1912] introduced this counting problem as a possible way to attack the Four Color Problem (Section 6.3).

In this section, we will discuss properties of the counting function, classes where it is easy to compute, and further related topics.

COUNTING PROPER COLORINGS

We start by defining the counting problem as a function of k.

5.3.1. Definition. Given $k \in \mathbb{N}$ and a graph G, the value $\chi(G; k)$ is the number of proper colorings $f\colon V(G) \to [k]$. The set of available colors is $[k] = \{1, \dots, k\}$; the k colors need not all be used in a coloring f. Changing the names of the colors that are used produces a different coloring.

5.3.2. Example. $\chi(\overline{K}_n; k)) = k^n$ *and* $\chi(K_n; k)) = k(k-1)\cdots(k-n+1)$.

When coloring the vertices of $\overline{K}_n$, we can use any of the k colors at each vertex no matter what colors we have used at other vertices. Each of the k^n functions from the vertex set to $[k]$ is a proper coloring, and hence $\chi(\overline{K}_n; k) = k^n$.

When we color the vertices of K_n, the colors chosen earlier cannot be used on the ith vertex. There remain $k-i+1$ choices for the color of the ith vertex no matter how the earlier colors were chosen. Hence $\chi(K_n; k) = k(k-1)\cdots(k-n+1)$.

5.3.23.* Definition. A **transitive orientation** of a graph G is an orientation D such that whenever xy and yz are edges in D, also there is an edge xz in G that is oriented from x to z in D. A simple graph G is a **comparability graph** if it has a transitive orientation.

5.3.24.* Example. If G is an X, Y-bigraph, then directing every edge from X to Y yields a transitive orientation. Thus every bipartite graph is a comparability graph. Transitive orientations arise from order relations; $x \to y$ could mean "x contains y", which is a transitive relation. ■

5.3.25.* Proposition. (Berge [1960]) Comparability graphs are perfect.

Proof: Every induced subdigraph of a transitive digraph is transitive, so the class of comparability graphs is hereditary. Thus we need only show that each comparability graph G is $\omega(G)$-colorable.

Let F be a transitive orientation of G; note that F has no cycle. As shown in proving Theorem 5.1.21, the coloring of G that assigns to each vertex v the number of vertices in the longest path of F ending at v is a proper coloring. By transitivity, the vertices of a path in F form a clique in G. Thus we have $\chi(G) \le \omega(G)$. ■

COUNTING ACYCLIC ORIENTATIONS (optional)

Surprisingly, $\chi(G; k)$ has meaning when k is a negative integer. An **acyclic orientation** of a graph is an orientation having no cycle. Setting $k = -1$ in $\chi(G; k)$ enables us to count the acyclic orientations of G.

5.3.26. Example. Since C_4 has 4 edges, it has 16 orientations. Of these, 14 are acyclic. In Example 5.3.7, we proved that $\chi(C_4; k) = k(k-1)(k^2 - 3k + 3)$. Evaluated at $k = -1$, this equals $(-1)(-2)(7) = 14$. ■

5.3.27. Theorem. (Stanley [1973]) The value of $\chi(G; k)$ at $k = -1$ is $(-1)^{n(G)}$ times the number of acyclic orientations of G.

Proof: We use induction on $e(G)$. Let $a(G)$ be the number of acyclic orientations of G. When G has no edges, $a(G) = 1$ and $\chi(G; -1) = (-1)^{n(G)}$, so the claim holds. We will prove that $a(G) = a(G-e) + a(G \cdot e)$ for $e \in E(G)$. If so, then we apply the recurrence for a, the induction hypothesis for $a(G)$ in terms of $\chi(G; k)$, and the chromatic recurrence to compute

$$a(G) = (-1)^{n(G)}\chi(G-e; -1) + (-1)^{n(G)-1}\chi(G \cdot e; -1) = (-1)^{n(G)}\chi(G; -1).$$

Now we prove the recurrence for a. Every acyclic orientation of G contains an acyclic orientation of $G - e$. An acyclic orientation D of $G - e$ may extend to 0, 1, or 2 acyclic orientations of G by orienting the edge $e = uv$. When D has no u, v-path, we can choose $v \to u$. When D has no v, u-path, we can choose $u \to v$. Since D is acyclic, D cannot have both a u, v-path and a v, u-path, so the two choices for e cannot both be forbidden.

Hence every D extends in at least one way, and $a(G)$ equals $a(G-e)$ plus the number of orientations that extend in both ways. Those extending in both ways are the acyclic orientations of $G - e$ with no u, v-path *and* no v, u-path. There are exactly $a(G \cdot e)$ of these, since a u, v-path or a v, u-path in an orientation of $G - e$ becomes a cycle in $G \cdot e$. ■

The interpretation of $\chi(G; k)$ for general negative k (Exercise 32) is an instance of the phenomenon of "combinatorial reciprocity" (Stanley [1974]).

EXERCISES

Keep in mind that the notation $\chi(G; k)$ may be viewed as a polynomial or as the number of proper k-colorings of G.

5.3.1. (–) Compute the chromatic polynomials of the graphs below.

5.3.2. (–) Use the chromatic recurrence to obtain the chromatic polynomial of every tree with n vertices.

5.3.3. (–) Prove that $k^4 - 4k^3 + 3k^2$ is not a chromatic polynomial.

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5.3.4. a) Prove that $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$.

b) For $H = G \vee K_1$, prove that $\chi(F; k) = k\chi(G; k-1)$. From this and part (a), find the chromatic polynomial of the wheel $C_n \vee K_1$.

5.3.5. For $n \ge 1$, let $G_n = P_n \,\square\, K_2$; this is the graph with $2n$ vertices and $3n - 2$ edges shown below. Prove that $\chi(G_n; k) = (k^2 - 3k + 3)^{n-1}k(k-1)$.

5.3.6. (!) Let G be a graph with n vertices. Use Proposition 5.3.4 to give a non-inductive proof that the coefficient of k^{n-1} in $\chi(G; k)$ is $-e(G)$.

5.3.7. Prove that the chromatic polynomial of an n-vertex graph has no real root larger than $n - 1$. (Hint: Use Proposition 5.3.4.)

5.3.8. (!) Prove that the number of proper k-colorings of a connected graph G is less than $k(k-1)^{n-1}$ if $k \ge 3$ and G is not a tree. What happens when $k = 2$?

5.3.9. (!) Prove that $\chi(G; x + y) = \sum_{U \subseteq V(G)} \chi(G[U]; x)\chi(G[\overline{U}]; y)$. (Hint: Since both sides are polynomials, it suffices to prove equality when x and y are positive integers; do this by counting proper $x + y$-colorings in a different way.)

5.3.10. Let G be a connected graph with $\chi(G;k) = \sum_{i=0}^{n-1}(-1)^i a_i k^{n-i}$. For $1 \le i \le n$, prove that $a_i \ge \binom{n-1}{i}$. (Hint: Use the chromatic recurrence.)

5.3.11. (!) Prove that the sum of the coefficients of $\chi(G;k)$ is 0 unless G has no edges. (Hint: When a function is a polynomial, how can one obtain the sum of the coefficients?)

5.3.12. (+) *Coefficients of* $\chi(G;k)$.

a) Prove that the last nonzero term in the chromatic polynomial of G is the term whose exponent is the number of components of G.

b) Use part (a) to prove that if $p(k) = k^n - ak^{n-1} + \cdots \pm ck^r$ and $a > \binom{n-r+1}{2}$, then p is not a chromatic polynomial. (For example, this immediately implies that the polynomial in Exercise 5.3.3 is not a chromatic polynomial.)

5.3.13. Let G and H be graphs, possibly overlapping.

a) Prove that $\chi(G \cup H; k) = \frac{\chi(G;k)\chi(H;k)}{\chi(G\cap H;k)}$ when $G \cap H$ is a complete graph.

b) Consider two paths whose union is a cycle to show that the formula may fail when $G \cap H$ is not a complete graph.

c) Apply part (a) to conclude that the chromatic number of a graph is the maximum of the chromatic numbers of its blocks.

5.3.14. (!) Let P be the Petersen graph. By Brooks' Theorem, the Petersen graph is 3-colorable, and hence by the pigeonhole principle it has an independent set S of size 4.

a) Prove that $P - S = 3K_2$.

b) Using part (a) and symmetry, determine the number of vertex partitions of P into three independent sets.

c) In general, how can the number of partitions into the minimum number of independent sets be obtained from the chromatic polynomial of G?

5.3.15. Prove that a graph with chromatic number k has at most k^{n-k} vertex partitions into k independent sets, with equality achieved only by $K_k + (n-k)K_1$ (a k-clique plus $n-k$ isolated vertices). (Hint: Use induction on n and consider the deletion of a single vertex.) (Tomescu [1971])

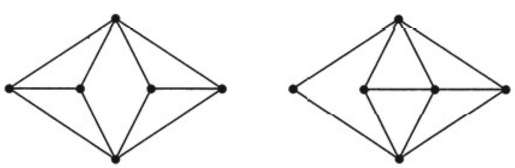
5.3.16. Let G be a simple graph with n vertices and m edges. Prove that G has at most $\frac{1}{3}\binom{m}{2}$ triangles. Conclude that the coefficient of k^{n-2} in $\chi(G;k)$ is positive, unless G has at most one edge. (Hint: Use Theorem 5.3.10.)

5.3.17. (∗) Use the inclusion-exclusion principle to prove Theorem 5.3.10 directly.

5.3.18. (!) Consider the chromatic polynomials of the graphs below.

a) Without computing them, give a short proof that they are equal.

b) Express this chromatic polynomial as the sum of the chromatic polynomials of two chordal graphs, and use this to give a one-line computation of it.



5.3.19. (−) Let G be the graph obtained from K_6 by subdividing one edge. Use the chromatic recurrence to Compute $\chi(G;k)$ as a product of linear factors (factors of the form $k - c_i$). Show that G is not a chordal graph. (Read [1975], Dmitriev [1980])

5.3.20. Let G be a chordal graph. Use a simplicial elimination ordering of G to prove the following statements.

a) G has at most n maximal cliques, with equality if and only if G has no edges. (Fulkerson–Gross [1965])

b) Every maximal clique of G containing no simplicial vertex of G is a separating set.

5.3.21. The **Szekeres–Wilf number** of a graph G is $1 + \max_{H\subseteq G}\delta(H)$. Prove that a graph G is chordal if and only if in every induced subgraph the Szekeres–Wilf number equals the clique number. (Voloshin [1982])

5.3.22. Let $k_r(G)$ be the number of r-cliques in a connected chordal graph G. Prove that $\sum_{r\ge1}(-1)^{r-1}k_r(G) = 1$. (Hint: Use induction on $n(G)$. Note that the binomial formula (Appendix A) implies that $\sum_{j\ge0}(-1)^j\binom{m}{j} = 0$ when $m \in \mathbb{N}$.)

5.3.23. Let S be the vertex set of a cycle in a chordal graph G. Prove that G has a cycle whose vertex set consists of all but one element of S. (Comment: When G has a spanning cycle and $S \subset V(G)$, Hendry conjectured that G also has a cycle whose vertex set consists of S plus one vertex.) (Hendry [1990])

5.3.24. Let e be a edge of a cycle C in a chordal graph. Prove that e forms a triangle with a third vertex of C.

5.3.25. Let Q be a maximal clique in a chordal graph G. Prove that if $G - Q$ is connected, then Q contains a simplicial vertex. (Voloshin–Gorgos [1982])

5.3.26. Exercise 5.3.13 establishes the formula $\chi(G \cup H; k) = \frac{\chi(G;k)\chi(H;k)}{\chi(G\cap H;k)}$ when $G \cap H$ is a complete graph.

a) Prove that the formula holds when $G \cup H$ is a chordal graph regardless of whether $G \cap H$ is a complete graph.

b) Prove that if x is a vertex in a chordal graph G, then

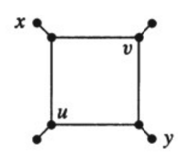
$$\chi(G;k) = \chi(G-x;k)k\frac{\chi(G[N(x)];k-1)}{\chi(G[N(x)];k)}.$$

(Comment: Part (b) allows the chromatic polynomial of a chordal graph to be computed via an arbitrary elimination ordering. For example, eliminating the central vertex of P_5 yields $\chi(P_5;k) = [k(k-1)]^2 k\frac{(k-1)^2}{k^2} = k(k-1)^4$.) (Voloshin [1982])

5.3.27. (+) A **minimal vertex separator** in a graph G is a set $S \subseteq V(G)$ that for some pair x, y is a minimal set whose deletion separates x and y. Every minimal separating set is a minimal vertex separator, but u, v below show that the converse need not hold.

a) Prove that if every minimal vertex separator in G is a clique, then the same property holds in every induced subgraph of G.

b) Prove that a graph G is chordal if and only if every minimal vertex separator is a clique. (Dirac [1961])



5.3.28. (!) Let G be an interval graph. Prove that G is a chordal graph and that $\overline{G}$ is a comparability graph.

5.3.29. Determine the smallest imperfect graph G such that $\chi(G) = \omega(G)$.

5.3.30. An edge in an acyclic orientation of G is **dependent** if reversing it yields a cycle.

a) Prove that every acyclic orientation of a connected n-vertex graph has at least $n-1$ independent edges.

b) Prove that if $\chi(G)$ is less than the girth of G, then G has an orientation with no dependent edges. (Hint: Use the technique in the proof of Theorem 5.1.21.)

5.3.31. (∗) The number $a(G)$ of acyclic orientations of G satisfies the recurrence $a(G) = a(G-e) + a(G \cdot e)$ (Theorem 5.3.27). The number of spanning trees of G appears to satisfy the same recurrence; does the number of acyclic orientations of G always equal the number of spanning trees? Why or why not?

5.3.32. (∗) Let D be an acyclic orientation of G, and let f be a coloring of $V(G)$ from the set $[k]$. We say that (D, f) is a **compatible pair** if $u \to v$ in D implies $f(u) \le f(v)$. Let $\eta(G; k)$ be the number of compatible pairs. Prove that $\eta(G; k) = (-1)^{n(G)}\chi(G; k)$. (Stanley [1973])

Chapter 6

Planar Graphs

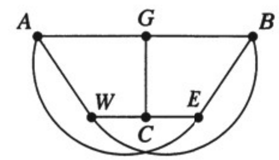
6.1. Embeddings and Euler's Formula

Topological graph theory, broadly conceived, is the study of graph layouts. Initial motivation involved the famous Four Color Problem: can the regions of every map on a globe be colored with four colors so that regions sharing a nontrivial boundary have different colors? Later motivation involves circuit layouts on silicon chips. Wire crossings cause problems in layouts, so we ask which circuits have layouts without crossings.

DRAWINGS IN THE PLANE

The following brain teaser appeared as early as Dudeney [1917].

6.1.1. Example. *Gas–water–electricity.* Three sworn enemies A, B, C live in houses in the woods. We must cut paths so that each has a path to each of three utilities, which by tradition are gas, water, and electricity. In order to avoid confrontations, we don't want any of the paths to cross. Can this be done? This asks whether $K_{3,3}$ can be drawn in the plane without edge crossings; we will give two proofs that it cannot. ■



Arguments about drawings of graphs in the plane are based on the fact that every closed curve in the plane separates the plane into two regions (the